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THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTION OF THE COVARIANCE EQUATION FOR A NAVIGATIONAL SYSTEM†

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An analytic solution of the equation for the covariances in a Kalman filter used to estimate the coordinates of an aircraft in a standard navigational system is presented and the long-term asymptotic forms of this solution are obtained.

In the general case the equations of a Kalman continuous linear filter [1] consist of the equations of the controlled process

$$dx/dt = A(t)x + u(t) + w_x, \quad x(0) = x_0 \tag{1}$$

the linear observation conditions

$$z = H(t)x + \upsilon(t)^{+} w_z \tag{2}$$

the equations

$$dy/dt = A(t)y + u(t) + RH^{\mathrm{T}}S^{-1}(z - Hy - \upsilon(t)), \quad y(0) = y_0 \tag{3}$$

for the estimate y for the state vector x, and the covariance equations

$$dR/dt = AR + RB + C - RDR \tag{4}$$

$$R(0) = R_0 = \mathrm{cov}(x_0, x_0) \tag{5}$$

In (1)–(5) $x = x(t)$ is the n-dimensional phase column-vector formed by the generalized coordinates of the controlled dynamical system, $z = z(t)$ is the p-dimensional column-vector that becomes known during the observation process (the observed vector), $y = y(t)$ is the n-dimensional column-vector estimating x computed from (3)–(5), $R = R(t) = \mathrm{cov}\,(x-y, x-y)$ is a symmetric $(n\times n)$-matrix computed from (4) with the initial condition (5), $A = A(t)$ and $H = H(t)$ are given continuous matrices of dimensions $n\times n$ and $p\times n$, respectively, such that $B = A^{\mathrm{T}}$, $u(t)$ and $\upsilon(t)$ are known time-dependent deterministic control functions appearing, respectively, in (1) and (2), x_0 is a Gaussian random vector with prescribed mean y_0 and covariance matrix R_0, and w_x and w_z are Gaussian white noise independent of one another and of x_0 with zero mean and with prescribed continuous correlation density matrices $C = C(t)$ and $S = S(t)$ (of dimensions $n\times n$ and $p\times p$, respectively)

$$\mathrm{cov}(w_x(t), w_x(\tau)) = C(t)\delta(t-\tau), \quad \mathrm{cov}(w_z(t), w_z(\tau)) = S(t)\delta(t-\tau)$$

$$\mathrm{cov}(w_x(t), x_0) = \mathrm{cov}(w_z(t), x_0) = \mathrm{cov}(w_x(t), w_z(\tau)) = 0$$

$$D = D(t) = H^{\mathrm{T}}(t)S^{-1}(t)H(t)$$

Everywhere above $t \geqslant 0$ and $\tau \geqslant 0$.

The analytic solution of Eq. (4), which is a Riccati matrix equation, enables us not only to solve (3) analytically (or numerically with smaller errors), but also to provide an accurate estimate of the potential precision of the controlled process. In particular, this applies to cases when the Kalman filter algorithm is applied to real dynamical systems for which the potential precision determines, in the end, whether

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or not it is economical to develop the system in nature. The precision is determined by the asymptotic behaviour (as $t \to +\infty$) of the matrix elements of R.

If a special solution $R = P(t)$ of (4) is known, then (4) can be reduced to a linear equation by the substitution $R = T + P$, $Q = T^{-1}$

$$dQ/dt = -QA_1 - B_1Q + D \quad (A_1 = A - PD, B_1 = A_1^{\mathrm{T}}) \tag{6}$$

If A, B, C and D are constant matrices and $P(t) \equiv P(0)$ is a stationary solution of (4), i.e. a solution of the algebraic Riccati matrix equation

$$AP + PB + C - PDP = 0 \tag{7}$$

then we make the substitution

$$Q = e^{-B_1t}Ge^{-A_1t} \Leftrightarrow T = e^{A_1t}G^{-1}e^{B_1t}$$

in (6), which converts (6) into the system

$$\dot{G} = e^{B_1t}De^{A_1t}$$

having the analytic solution

$$\begin{aligned} Q(t) &= e^{-B_1t}\left\{G(0) + \int_0^t e^{B_1s}De^{A_1s}ds\right\}e^{-A_1t} \\ G(0) &= Q(0) = (R_0 - P)^{-1} \\ R(t) &= P + e^{A_1t}\left\{G(0) + \int_0^t e^{B_1s}De^{A_1s}ds\right\}^{-1} e^{B_1t} \end{aligned} \tag{8}$$

We shall use this method to find an analytic solution of the covariance equation for a number of standard navigational systems [2] which use the Kalman filtration algorithm.

Problem 1 (Four-dimensional filter). For the most widely used version of the construction of a navigational system consisting of one inertial navigational system, in the cyclic algorithm for estimating the parameters of vertical motion of an aircraft using an aneroid altimeter and a radio altimeter in the case when the vertical accelerometer signal is used as the control interaction, the observation and control conditions of the controlled process [2, p. 160] have matrices

$$\begin{aligned} &A = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}, \quad H = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \\ &C = \mathrm{diag}(0, c^2, 0, 0), \quad S = \mathrm{diag}(a^{-2}, b^{-2}) \\ &D = \mathrm{diag}(a^2, 0, 0, b^2), \quad R_0 = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2) \end{aligned} \tag{9}$$

where $a > 0, b > 0, c > 0, \sigma_i > 0$ $(i = 1, \ldots, 4)$ are constants.

The solution of the matrix equation (7) yields [3]

$$P = \begin{Vmatrix} \sqrt{2c/a^3} & c/a & 0 & 0 \\ c/a & \sqrt{2c^3/a} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix} \tag{10}$$

Equation (6) for the components θ_{ij} $(i, j = 1, \ldots, 4)$ of the symmetric matrix Q takes the form

$$\dot{\theta}_{11} = 4\theta\theta_{11} + 4\theta^2\theta_{12} + a^2, \quad \dot{\theta}_{22} = -2\theta_{12}, \quad \dot{\theta}_{12} = -\theta_{11} + 2\theta\theta_{12} + 2\theta^2\theta_{22}$$
$$\dot{\theta}_{14} = 2\theta\theta_{14} + 2\theta^2\theta_{24}, \quad \dot{\theta}_{24} = -\theta_{14}, \quad \dot{\theta}_{13} = 2\theta\theta_{13} + 2\theta^2\theta_{23} + \theta_{12} \tag{11}$$
$$\dot{\theta}_{23} = -\theta_{13} + \theta_{22}, \quad \dot{\theta}_{33} = 2\theta_{23}, \quad \dot{\theta}_{34} = \theta_{24}, \quad \dot{\theta}_{44} = b^2$$

where

$$\theta = \sqrt{ac/2} \tag{12}$$

The solutions of (11) are given by the following formulae as functions of the dimensionless time parameter $\tau = \theta t$

$$\theta_{11}(t) = (\alpha_1/2)[(C_1 - C_2\sin 2\tau + C_3\cos 2\tau)e^{2\tau} - 1]$$
$$\theta_{12}(t) = (\alpha_2/2)e^{2\tau}[-C_1 + (C_2 - C_3)\sin 2\tau - (C_2 + C_3)\cos 2\tau]$$
$$\theta_{22}(t) = \alpha_3[(C_1 + C_2\cos 2\tau + C_3\sin 2\tau)e^{2\tau} - 1]$$
$$\theta_{13}(t) = \alpha_3[e^{\tau}(D_1\cos\tau + D_2\sin\tau) - 1 - (C_3\sin 2\tau + C_2\cos 2\tau)e^{2\tau}]$$
$$\theta_{23}(t) = \alpha_4[e^{\tau}(-(D_1 + D_2)\sin\tau + (D_2 - D_1)\cos\tau) + \\ +2 + e^{2\tau}(C_1 + (C_2 + C_3)\sin 2\tau + (C_2 - C_3)\cos 2\tau)]$$
$$\theta_{33}(t) = 2\alpha_5[-2D_2 + C_3 - C_1 + 2e^{\tau}(-D_1\sin\tau + D_2\cos\tau) + \\ +4\tau + e^{2\tau}(C_1 + C_2\sin 2\tau - C_3\cos 2\tau)] + \sigma_3^{-2}$$
$$\theta_{44}(t) = (b^2/\theta)(\tau + \eta), \quad \theta_{14} = \theta_{24} = \theta_{34} \equiv 0 \tag{13}$$

Taking (5) and (9) into account, we set

$$\alpha_k = a^2/(2\theta)^k, \quad k = 1,\ldots,5; \quad \Delta_1 = 2\sigma_2^2\alpha_3 - 1, \quad \Delta_2 = \sigma_1^2\alpha_1 - 1$$
$$\Delta_0 = 2\Delta_1\Delta_2 - 1, \quad C_1 = 4(\Delta_1 + \Delta_2 + 1)/\Delta_0 + 2$$
$$C_2 = -1 - 4(\Delta_1 + 1)/\Delta_0, \quad C_3 = -1 - 4(\Delta_2 + 1)/\Delta_0$$
$$D_1 = C_2 + 1, \quad D_2 = C_3 - C_1 - 1, \quad \eta = \theta/(b^2\sigma_4^2)$$

in (12). Finally, we have

$$R = \left\| \begin{array}{ccc|c} & \omega & & 0 \\ & & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & r_{44} \end{array} \right\|, \quad \omega = \Omega^{-1} + \left\| \begin{matrix} p_{11} & p_{12} & 0 \\ p_{12} & p_{22} & 0 \\ 0 & 0 & 0 \end{matrix} \right\|, \quad \Omega = \left\| \begin{matrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{12} & \theta_{22} & \theta_{23} \\ \theta_{13} & \theta_{23} & \theta_{33} \end{matrix} \right\| \tag{14}$$

where

$$p_{11} = 2\theta/a^2, \quad p_{22} = 4\theta^3/a^2, \quad p_{12} = 2\theta^2/a^2, \quad r_{44} = \theta/(b^2(\tau + \eta)) \tag{15}$$

The evaluation of the inverse matrix in (14) involves finding $\det\Omega = q$, for which, using (11), we obtain the following equation by differentiation

$$dq/d\tau = 4q + 2\alpha_1(\theta_{22}\theta_{33} - \theta_{23}^2) \tag{16}$$

From (13) it follows that

$$\theta_{22}\theta_{33} - \theta_{23}^2 = 2\alpha_4^2[(C_1 + C_2\cos 2\tau + C_3\sin 2\tau)e^{2\tau} - 1]\times \\ \times[2e^{\tau}(-D_1\sin\tau + D_2\cos\tau) + 4\tau + e^{2\tau}(C_1 + C_2\sin 2\tau - C_3\cos 2\tau) + \\ +1 - D_2 + \sigma_3^{-2}/(2\alpha_5)] - \alpha_4^2[e^{\tau}(-(D_1 + D_2)\sin\tau + (D_2 - D_1)\cos\tau) + \\ +2 + e^{2\tau}(C_1 + (C_2 + C_3)\sin 2\tau + (C_2 - C_3)\cos 2\tau)]^2 \tag{17}$$

We shall find the asymptotic behaviour (as $t \to +\infty$) of the elements of the matrix R. The main term in (17) has exponential growth and equals

$$2\alpha_4^2 e^{4\tau}[4(\Delta_1 + \Delta_2 + 2)/\Delta_0 + 1] = \Delta e^{4\tau}$$

This is the only resonance term on the right-hand side of (16). Consequently, for $\Delta > 0$ (this assumption will henceforth be retained)

$$q(\tau) \sim (q(0) + 2\alpha_1 \Delta\tau)e^{4\tau}, \quad \tau \to +\infty$$

and

$$|\omega_{11} - p_{11}| \sim \theta/(a^2\tau) = 1/(a^2 t), \quad t \to +\infty \tag{18}$$

By analogy

$$\theta_{11}\theta_{33} - \theta_{13}^2 = 4\theta^2 \Delta e^{4\tau} + \ldots$$

in connection with which

$$|\omega_{22} - p_{22}| \sim 4\theta^3/(a^2\tau) = 4\theta^2/(a^2 t), \quad t \to +\infty \tag{19}$$

Furthermore

$$|\omega_{12} - p_{12}| \sim 2\theta^2/(a^2\tau) = 2\theta/(a^2 t), \quad t \to +\infty \tag{20}$$

Formulae (18)–(20) determine the asymptotic behaviour of the matrix elements

$$r_{11} = \omega_{11}, \quad r_{12} = \omega_{12}, \quad r_{22} = \omega_{22}$$

of R for large t. Moreover, according to these formulae, a characteristic criterion for t to be "large" is that the ratios

$$(a^2 t)^{-1} : p_{11} = \frac{1}{2\tau}, \quad 4\theta^2(a^2 t)^{-1} : p_{22} = \frac{1}{\tau}, \quad 2\theta(a^2 t)^{-1} : p_{12} = \frac{1}{\tau}$$

should be small. Therefore, t should be considered large if

$$\frac{1}{\tau} \ll 1 \Leftrightarrow t \gg \frac{1}{\theta}$$

We find the asymptotic behaviour of the other elements of R

$$|r_{13}| = |(\theta_{12}\theta_{23} - \theta_{22}\theta_{13})/q| \sim 2\theta^2/(a^2 t)$$

$$|r_{23}| = |(\theta_{12}\theta_{13} - \theta_{11}\theta_{23})/q| \sim 4\theta^3/(a^2 t)$$

$$|r_{33}| \sim 4\theta^4/(a^2 t); \quad |r_{44}| \sim 1/(b^2 t), \quad t \to +\infty \tag{21}$$

Problem 2 (Three-dimensional filter). During a flight over very rugged terrain and in the mountains substantial errors occur in the radio altimeter channel either in the form of the high-frequency component of the terrain field or as errors due to the inconsistency between the realization of the field transducer and the realization obtained from the memory unit according to the signals from the navigational system. In this case it is necessary [2, p. 262] to change from a four-dimensional filter to a three-dimensional one, which is connected with the deviation of the radio altimeter signal. In general notation, this corresponds to the matrices

$$A = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{Vmatrix}, \quad \begin{matrix} H = \|1, 0, 0\| \\ S = \|a^{-2}\| \end{matrix}$$

$$C = \mathrm{diag}(0, c^2, 0), \quad D = \mathrm{diag}(a^2, 0, 0), \quad R_0 = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2)$$

The solution of the matrix equation (7) has the form (10) with the last row and the last column removed.

The matrix $R = R(t)$ is equal to the matrix ω outlined in (14) and defined by (15). It follows that all the formulae and asymptotic expressions obtained above hold for R.

It is interesting to compare the derivation of (18)–(21) with the results of [4] for the discrete version of the problem (the latter being given in the braces)

$$r_{11} \sim \frac{2\theta}{a^2} + \frac{1}{a^2 t} \quad \left\{\sim \frac{9}{a^2 t}\right\}$$
$$r_{22} \sim \frac{4\theta^3}{a^2} + \frac{4\theta^2}{a^2 t} \quad \left\{\sim \frac{192}{a^2 t^3}\right\}$$

Similar differences occur in the asymptotic expressions for other terms of $R(t)$ too. They are connected with the fact that the simplifying assumption $D_x \to 0$ [2, p. 265], which distorts the time scale, is made in the solution in [4].

Problem 3 (Two-dimensional filter). The consideration of a navigational system consisting of all three channels of an inertial navigational system, which measure the coordinates of the spatial and angular position of the aircraft, a radio altimeter and an on-board computer with a ground topography map stored in the memory (barometric instruments are not included) leads [2, p. 222] to problem (1)–(5) with

$$A = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \quad H = \|1, 0\|, \quad S = \|a^{-2}\|$$
$$C = \mathrm{diag}(0, c^2), \quad D = \mathrm{diag}(a^2, 0), \quad R_0 = \mathrm{diag}(\sigma_1^2, \sigma_2^2)$$

The model of errors in determining the coordinate and velocity of a horizontally flying aircraft using an inertial navigational system leads to the same problem [2, p. 159].

The problem was studied [2] under assumptions which distort the time scale. In the present paper exact solutions and asymptotic approximations are obtained.

For the problem under consideration (6) takes the form of the first three equations in (11), and its solution is given by the first three formulae in (13). For such a choice of functions

$$q = \theta_{11}\theta_{22} - \theta_{12}^2 = \alpha_2^2 q^* / 4$$
$$q^* = (C_1^2 - C_2^2 - C_3^2)e^{4\tau} - 2(2C_1 + (C_2 + C_3)\cos 2\tau + (C_3 - C_2)\sin 2\tau)e^{2\tau} + 2$$

The resulting formulae yield an exponentially decaying asymptotic forms for the matrix elements r_{ij}

$$r_{11} = p_{11} + \theta_{22} / q = \frac{2\theta}{a^2}(1 + O(e^{-2\tau})), \quad r_{22} = \frac{4\theta^3}{a^2}(1 + O(e^{-2\tau}))$$
$$r_{12} = \frac{2\theta^2}{a^2}(1 + O(e^{-2\tau})), \quad \tau \to +\infty$$

as opposed to the power function behaviour in formulae (5.103) of [2] obtained under the assumption $S_j = 0$, which distorts the timescale (see [2, p. 162]).

We remark that linear models of Kalman filtering usually provide an adequate description of the actual processes involving the observation of non-linear objects only within bounded time intervals. Therefore the results of this paper can only be applied once the duration of these intervals has been compared with the characteristic time constant introduced in (12).

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